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Flow and heat transfer of an electrically conducting fluid of second grade over a stretching sheet subject to a transverse magnetic field

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Abstract

The paper presents analytical solutions for the flow and heat transfer in a steady laminar boundary flow of an electrically conducting fluid of second grade subject to a transverse uniform magnetic field past a semi-infinite stretching sheet with power-law surface temperature or power-law surface heat flux. The effects of viscous dissipation, internal heat generation or absorption, work done due to deformation and Joule heating are considered in the energy equation. The variations of surface temperature gradient for the prescribed surface temperature (PST case) and surface temperature for the prescribed surface heat flux (PHF case) with various parameters are graphed and tabulated. Asymptotic solutions of the temperature for large Prandtl number are also given for two heating conditions. The inclusion of the Joule heating has a significant influence on the thermal characteristics at the wall especially when the Eckert number, magnetic parameter as well as the Prandtl number are large. When the Eckert number is large enough, the heat may flow from the fluid region to the wall in contrast to that when Eckert number is small. A physical explanation is proposed for this phenomenon.

Keywords: Second grade fluid; Stretching sheet; Boundary layer flow; Magnetic field

1. Introduction

The study of laminar boundary layer flow over a stretching sheet has received considerable attention in the past, for example, materials manufactured by extrusion process and heat-treated materials traveling between a feed roll and a wind-up roll or on conveyor belts possess the features of a moving continuous surface. In view of these applications, Sakiadis [1] initiated the study of boundary layer flow over a continuous solid surface moving with constant speed and then extended to stretching sheet by Crane [2]. Following them, Gupta and Gupta [3] examined the heat and mass transfer using a similarity transformation for the boundary layer flow over a stretching sheet subject to suction or blowing. The effects of power law surface temperature and power law surface heat flux on the heat transfer characteristics of a continuous, stretching surface with suction or blowing were investigated by Chen and Char [4]. However, above researches are restricted to flows of Newtonian fluids.

Many materials such as polymer solutions or melts, drilling mud, clastomers, certain oils and greases and many other emulsions are classified as non-Newtonian fluids. There are many models describing the properties, but not all, of non-Newtonian fluids. These models or constitutive equations, however, cannot describe all the behaviors of these non-Newtonian fluids, for example, normal stress differences, shearing thinning or shearing thickening, stress relaxation, elastic effects and memory

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Nomenclature

$(a)_n$	symbol in Eq. (22)
A	proportional constant
$\mathbf{A}_1$, $\mathbf{A}_2$	first and second Rivlin–Ericksen tensors
$(b)_n$	symbol in Eq. (22)
B	proportional constant
B_0	component of magnetic field
$\mathbf{B}$	magnetic field vector
c_p	specific heat at constant pressure
c_f	skin friction coefficient
D	proportional constant
Ec	Eckert number
f	dimensionless velocity
g	dimensionless temperature in PHF case
k	thermal conductivity
K	viscoelastic parameter
l	characteristic length
$\mathbf{L}$	$=\nabla\mathbf{V}$
m	combined parameter
Mn	magnetic parameter
$M(a,b,z)$	Kummer's function
Nu	Nusselt number
p	pressure/parameter in Eq. (22)
Pr	Prandtl number
q	internal heat source or absorption
Q	parameter in Eq. (22)
r	parameter
Re	Reynolds number
s	parameter in Eq. (22)
T	temperature
T_∞	ambient temperature
T_w	wall temperature
$\mathbf{T}$	Cauchy stress tensor
u, v	velocity component in the x and y directions
$\mathbf{V}$	velocity vector
x, y	coordinates

Greek symbols

α	dimensionless internal parameter
α_1, α_2	material constants
δ_T	thermal boundary layer thickness
ϕ	dimensionless temperature
Φ	dimensionless temperature
η	dimensionless coordinate
μ	dynamic viscosity
ν	kinematic viscosity
θ	dimensionless temperature in PST case
ρ	density
ξ	dimensionless coordinate
ζ	dimensionless coordinate

effects, etc. Among these models, the fluid of differential type, for example, fluids of second grade and third grade, have been received much attention in the past due to their elegance and simplicity [5].

Recently, the studies of boundary layer flow of non-Newtonian fluids over a stretching sheet become more importantly because of industrial applications. Fox et al. [6] used both exact and approximate methods to examine the boundary layer flow of a viscoelastic fluid characterized by a power law model. Vajravelu and Rollins [7] investigated the heat transfer of the boundary layer flow of a second grade fluid whose constitutive equation is given by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{1}$$

Here $\mathbf{T}$ is the Cauchy stress tensor, p is the indeterminate pressure constrained by the incompressibility, μ is the viscosity, α_1 and α_2 are the moduli of the viscoelastic fluid, and $\mathbf{A}_1$ and $\mathbf{A}_2$ are the first two Rivlin–Ericksen tensors defined by [8]

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^{\mathrm{T}},$$

$$\mathbf{A}_2 = \frac{\mathrm{d}\mathbf{A}_1}{\mathrm{d}t} + \mathbf{A}_1\mathbf{L} + \mathbf{L}^{\mathrm{T}}\mathbf{A}_1, \tag{2}$$

where $\mathrm{d}/\mathrm{d}t$ is the material derivative and $\mathbf{L} = \nabla\mathbf{V}$. If the fluid of second grade is to satisfy the Clausius–Dehum inequality for all motions and the assumption that the specific Helmholtz free energy of the fluid is a minimum when it is locally at rest, then the requirements for the moduli of the second grade fluid are

$$\mu \geqslant 0, \quad \alpha_1 > 0 \quad \text{and} \quad \alpha_1 + \alpha_2 = 0. \tag{3}$$

Though the sign of α_1 has been a subject of much controversy. We do not intend to discuss it since a critical review of Dunn and Rajagopal [5] has already given a concise discussion about this issue. The fluids of second grade with negative α_1 may result in a physically impossible flow situation, which is not compatible with the stability criteria, however, solutions exist for a variety of flows when α_1 is taken to be positive. Vajravelu and Rollins [7] used the negative sign for α_1 in the stretching problem of a viscoelastic fluid, the Walters' B liquid, without considering the deformation work in the energy equation, however, a positive α_1 is chosen for the same problem of a second grade fluid including the work done due to deformation [9].

If the second grade fluid is electrically conducting, the Lorentz force $\mathbf{J} \times \mathbf{B}$, where $\mathbf{J}$ is the electrical current and

B is the magnetic field, must be included in the momentum equation when a transverse uniform magnetic field $\mathbf{B} = (0, B_0, 0)$ is applied to the fluid layer. The terms due to Lorentz force can be simplified if the following assumptions are made: (i) All physical quantities are constant; (ii) the magnetic field **B** is perpendicular to the velocity **V** and the induced magnetic field is small compared with the applied magnetic field; (iii) the electrical field is assumed to be zero. These assumptions are valid when the magnetic Reynolds number is small and there is no displacement current [10]. Thus, in the boundary layer approximation the Lorentz force is simply the term $-\sigma B_0^2 u$, where σ is the electrical conductivity, B_0 is the uniform magnetic field in the y-direction, and u is the x-component of velocity **V**. The flow problem of non-Newtonian fluids, characterized by Bingham plastic and the power law models, in a magnetic field has been investigated by Sarpkaya [11]. Sarpkaya also pointed out that some non-Newtonian fluids such as nuclear fuel slurries, liquid metals, mercury amalgams, biological fluids, plastic extrusions, paper coating, lubrication oils and greases, have applications in many areas in the absence as well as in the presence of a magnetic field. Char [12] studied the heat and mass transfer in a hydromagnetic flow of a visco-elastic fluid, the Walters' B liquid, over a stretching sheet, however, only the thermal diffusion is considered in the energy equation.

Motivated by the possible industrial applications and previous studies regarding the flow and heat transfer of non-Newtonian fluids over the stretching sheet, we present analytical solutions for flow and heat transfer of a laminar boundary layer flow of an electrically conducting second grade fluid subject to a transverse uniform magnetic field over a stretching sheet with prescribed power-law surface temperature and prescribed power-law surface heat flux. Here the viscoelastic modulus α_1 of the second grade fluid is taken to be positive to satisfy thermodynamic restrictions Eq. (3). The energy equation we considered includes the viscous dissipation, work due to deformation, internal heat generation or absorption, and the Joule heating. Although Sarma and Rao [13] solved the relevant problem analytically for a visco-elastic fluid with $\alpha_1 < 0$, they do not consider an electrically conducting fluid so as not to include the effects of the magnetic field, for example, the Lorentz force in the momentum equation and the Joule heating in the energy equation. Here we adopt their solutions only with minor modification to solve the problem in which we are interested. Also asymptotic solutions for the temperature function are given when the Prandtl number is large. The effects of the inclusion of the Joule heating, Eckert number and the Prandtl number on the thermal characteristics at the wall are examined in details.

2. Formulation and solutions

Consider an incompressible, electrically conducting fluid of second-grade, obeying Eqs. (1) and (2) with $\alpha_1 > 0$, subject to a transverse uniform magnetic field over a semi-infinite stretching sheet with the plane $y = 0$, then the fluid occupies above the sheet $y > 0$. Two equal and opposite forces are introduced along the x-axis so that the sheet is stretched keeping the origin fixed. In the assumptions of boundary layer flow, the governing equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{4}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + \frac{\alpha_1}{\rho}\left[\frac{\partial}{\partial x}\left(u\frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^3 u}{\partial y^3}\right] - \frac{\sigma B_0^2}{\rho}u, \tag{5}$$

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2} + \mu\left(\frac{\partial u}{\partial y}\right)^2 + \alpha_1\frac{\partial u}{\partial y}\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right) + q(T - T_\infty) + \sigma B_0^2 u^2, \tag{6}$$

where u and v are velocity components, T is the temperature, T_∞ is the temperature of the ambient fluid, ρ is the density, q is the specific heat generation rate, $\nu = \mu/\rho$ is kinematic viscosity, k is the conductivity and c_p is the specific heat at constant pressure. In deriving (5) and (6) it is assumed that the contribution due to the normal stress is of the same order of magnitude as that due to the shear stress. The last term in (5) is the Lorentz force and the last three terms in (6) are work done due to deformation, internal heat generation or absorption and the Joule heating. We assumed that the gravity force is neglected and the modified pressure gradient is absent since the flow is driven by the stretching sheet.

The appropriate boundary conditions for velocity field are

$$u = Bx, \quad v = 0 \qquad \text{at } y = 0,\ B > 0,$$
$$u \to 0, \quad \partial u/\partial y \to 0 \qquad \text{at } y \to \infty. \tag{7}$$

It has been implicitly assumed that the diffusion rate at the stretching sheet results in a negligible normal velocity v. Besides, the condition $\partial u/\partial y \to 0$ at $y \to \infty$, is the augmented condition, which has been discussed by Garg and Rajagopal [14], solving a flow problem via a singular perturbation technique.

The thermal boundary conditions for the energy equation (6) are

$$\text{PST case:} \quad T = T_w = T_\infty + A\left(\frac{x}{l}\right)^2 \quad \text{at } y = 0,$$

$$\text{PHF case:}\quad q_w = -k\frac{\partial T}{\partial y} = D\left(\frac{x}{l}\right)^2 \quad \text{at } y=0,$$

$$T \to T_\infty \quad \text{as } y \to \infty, \tag{8}$$

where A and D are constants, l is the characteristic length, q_w is the wall heat flux.

A similarity solution for velocity exists if we introduce a transformation

$$u = Bxf'(\eta), \quad v = -(B\nu)^{1/2}f(\eta), \quad \eta = (B/\nu)^{1/2}y, \tag{9}$$

where a prime denotes the differentiation with respect to η. Apparently (9) has already satisfied the continuity equation (4). Substituting (9) into (5), we have

$$f'^2 - ff'' = f''' + K(2f'f''' - f''^2 - ff^{IV}) - \mathrm{Mn}f', \tag{10}$$

where $K = \alpha_1 B/\mu$ is the viscoelastic parameter and $\mathrm{Mn} = \sigma B_0^2/\rho B$ is the magnetic parameter. When $\mathrm{Mn} = 0$, (10) reduces to the problem without the magnetic field. We note that (10) is exactly the same as that of (2) in Subhas and Veena [15], which discussed a saturated viscoelastic fluid of porous medium over a stretching sheet, provided that K and Mn are replaced by $-K_1$, and K_2, respectively.

The corresponding boundary conditions (7) become

$$\begin{aligned} &f = 0, \quad f' = 1 \qquad \text{at } \eta = 0,\\ &f' \to 0, \quad f'' \to 0, \qquad \text{at } \eta \to \infty. \end{aligned} \tag{11}$$

An exact solution to (10) and (11), following the procedure of Troy et al. [16], is obtained as

$$f(\eta) = \frac{1}{m}(1 - \mathrm{e}^{-m\eta}) \tag{12}$$

where $m = \sqrt{(1+\mathrm{Mn})/(1+K)}$ is a combined parameter relating the effects of viscoelasticity of the second grade fluid and the magnetic field.

The velocity components are given by

$$\begin{aligned} &u = Bx\mathrm{e}^{-m\eta}\\ &v = -(B\nu)^{1/2}(1 - \mathrm{e}^{-m\eta})/m. \end{aligned} \tag{13}$$

Since we have discarded the possibility of negative value of α_1, thus the solution of exponential type (12), corresponding to positive α_1 or K, is the only solution which is physically possible, rather than the new solution discussed by Chang et al. [17].

The dimensionless shear stress at the stretching sheet $\eta = 0$ is characterized by the skin friction coefficient c_f as

$$c_f = \frac{T_{12}(0)}{\rho u_w^2/2} = \frac{2}{Re^{1/2}}(1+3K)f''(0), \tag{14}$$

where $Re = Bx^2/\nu$ is the local Reynolds number and the dimensionless velocity gradient at the wall is given by

$$f''(0) = -m. \tag{15}$$

2.1. The prescribed surface temperature (PST case)

If we introduce the dimensionless temperature $\theta(\eta)$ in the PST case as

$$\theta(\eta) = \frac{T - T_\infty}{T_w - T_\infty}, \tag{16}$$

and use (9), then the energy equation (6) becomes

$$\begin{aligned} &\theta'' + Prf\theta' - Pr(2f' - \alpha)\theta\\ &\quad = -PrEc[(f'')^2 + Kf''(f'f'' - ff''') + \mathrm{Mn}(f')^2], \end{aligned} \tag{17}$$

where a prime denotes differentiation with respect to η, $Pr = \mu c_p/k$ is the Prandtl number, $\alpha = q/B\rho c_p$ is the internal heat parameter and $Ec = B^2l^2/Ac_p$ is the Eckert number.

The corresponding thermal boundary conditions are

$$\begin{aligned} &\theta = 1 \quad \text{at } \eta = 0,\\ &\theta \to 0 \quad \text{as } \eta \to \infty. \end{aligned} \tag{18}$$

Substituting (12) into (17) and (18) and introducing the transformation $\xi = -r\mathrm{e}^{-m\eta}$ with $r = Pr/m^2$, we have

$$\begin{aligned} &\xi\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} + (1 - r - \xi)\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + \left(2 + \frac{\alpha r}{\xi}\right)\theta\\ &\quad = -PrEc\left[1 + K + \frac{\mathrm{Mn}}{m^2}\right]\frac{\xi}{r^2}, \end{aligned} \tag{19}$$

and

$$\theta(-r) = 1 \quad \text{and} \quad \theta(0^-) = 0. \tag{20}$$

Eq. (19) can be further transformed into the standard confluent hypergeometric equation or the Kummer's equation [18], the solution satisfies (19) and (20) is given by

$$\theta(\xi) = (1+Q)\left(\frac{\xi}{-r}\right)^p \frac{M(p-2, s+1, \xi)}{M(p-2, s+1, -r)} - Q\left(\frac{\xi}{-r}\right)^2, \tag{21}$$

where

$$s = \sqrt{r^2 - 4\alpha r}, \quad p = (r+s)/2,$$

$$Q = \frac{EcPr[1 + K + \mathrm{Mn}/m^2]}{4 - 2r + \alpha r}$$

and

$$M(a,b,z) = 1 + \sum_{n=1}^{\infty}\frac{(a)_n}{(b)_n}\frac{z^n}{n!} \text{ is the Kummer's function,}$$

$$\begin{aligned} &(a)_n = a(a+1)(a+2)\cdots(a+n-1),\\ &(b)_n = b(b+1)(b+2)\cdots(b+n-1). \end{aligned} \tag{22}$$

When the condition $r \to 4/(2-\alpha)$ is met, the particular solution in (21) becomes invalid, thus we exclude this possibility for simplicity. However, the solution in this situation can be found in Sarma and Rao [13].

The solution (21), in terms of η, is

$$\theta(\eta) = (1+Q)\mathrm{e}^{-pm\eta}\frac{M(p-2, s+1, -r\mathrm{e}^{-m\eta})}{M(p-2, s+1, -r)} - Q\mathrm{e}^{-2m\eta}, \tag{23}$$

and the dimensionless surface temperature gradient at the wall is

$$\theta'(0) = (1+Q)m\left[r\frac{p-2}{(s+1)}\frac{M(p-1, s+2, -r)}{M(p-2, s+1, -r)} - p\right] + 2mQ. \tag{24}$$

The local heat transfer rate at the wall is characterized by the Nusselt number Nu as

$$Nu = \frac{-k\frac{\partial T}{\partial y}\big|_{y=0}}{k(T_\mathrm{w} - T_\infty)}x = -Re^{1/2}\theta'(0). \tag{25}$$

2.2. The prescribed surface heat flux (PHF case)

For the prescribed surface heat flux (PHF) case, the dimensionless temperature is defined as

$$T - T_\infty = \frac{D}{k}\left(\frac{x}{l}\right)^2\left(\frac{v}{B}\right)^{1/2} g(\eta), \tag{26}$$

and the corresponding energy equation becomes

$$g'' + Pr\,fg' - Pr(2f' - \alpha)g = -Pr\,Ec[f''^2 + Kf''(f'f'' - ff''') + \mathrm{Mn}(f')^2], \tag{27}$$

where $Ec = kB^2l^2(B/v)^{1/2}/Dc_p$, which is different from the Eckert number in the PST case and all other parameters are the same as before.

Using the same transform $\xi = -r\mathrm{e}^{-m\eta}$ and (12), the energy equation (27) has the same form as (19) and the boundary conditions (8b) and (8c) now become

$$g'(-r) = \frac{-1}{rm} \quad \text{and} \quad g(0^-) = 0. \tag{28}$$

The solution satisfies (27) and (28) is given by

$$g(\xi) = \left(\frac{1}{m} + 2Q\right)\left(\frac{\xi}{-r}\right)^p \left[pM(p-2, s+1, -r) - r\frac{p-2}{s+1}M(p-1, s+2, -r)\right]^{-1} M(p-2, s+1, \xi) - Q\left(\frac{\xi}{-r}\right)^2, \tag{29}$$

or in terms of η as

$$g(\eta) = -Q\mathrm{e}^{-2m\eta} + \left(\frac{1}{m} + 2Q\right)\mathrm{e}^{-pm\eta}\left[pM(p-2, s+1, -r) - r\frac{p-2}{s+1}M(p-1, s+2, -r)\right]^{-1} \times M(p-2, s+1, -r\mathrm{e}^{-m\eta}). \tag{30}$$

The wall temperature can be obtained from (26) as

$$T_\mathrm{w} - T_\infty = \frac{D}{k}\left(\frac{x}{l}\right)^2\left(\frac{v}{B}\right)^{1/2} g(0), \tag{31}$$

where

$$g(0) = -Q + \left(\frac{1}{m} + 2Q\right)M(p-2, s+1, -r) \times \left[pM(p-2, s+1, -r) - \frac{r(p-2)}{s+1} \times M(p-2, s+1, -r)\right]^{-1}, \tag{32}$$

is the dimensionless wall temperature.

3. Asymptotic analysis

We examine the asymptotic behavior of solutions of the energy equation when Pr is very large for both the PST and PHF cases,

(i) The PST case

In the limit $Pr \to \infty$, the energy equation has the form

$$\theta'' + \frac{Pr}{m}(1 - \mathrm{e}^{-m\eta})\theta' - Pr(2\mathrm{e}^{-m\eta} - \alpha)\theta = -Pr\,Ec\left(1 + K + \frac{\mathrm{Mn}}{m^2}\right)m^2\mathrm{e}^{-2m\eta}. \tag{33}$$

Letting $\phi = \theta + Q\mathrm{e}^{-2m\eta}$, we have

$$\frac{1}{Pr}\phi'' + \frac{1}{m}(1 - \mathrm{e}^{-m\eta})\phi' - (2\mathrm{e}^{-m\eta} - \alpha)\phi = 0. \tag{34}$$

The corresponding boundary conditions are

$$\phi = 1 + Q \quad \text{at } \eta = 0, \qquad \phi \to 0 \quad \text{as } \eta \to \infty. \tag{35}$$

Since the thermal boundary layer thickness δ_T is of the order $1/\sqrt{Re}\sqrt{Pr}$, we introduce the coordinate transformation $\zeta = \sqrt{Pr}\eta$ and let $\Phi = \mathrm{e}^{-\frac{1}{4}\zeta^2}\phi$, then the energy equation (34) becomes

$$\frac{\mathrm{d}^2\Phi}{\mathrm{d}\zeta^2} - \left(\frac{1}{4}\zeta^2 + \frac{5}{2} - \alpha\right)\Phi = 0, \tag{36}$$

and the corresponding boundary conditions are

$$\Phi = 1 + Q \quad \text{at } \zeta = 0, \qquad \Phi \to 0 \quad \text{as } \zeta \to \infty. \tag{37}$$

In obtaining (36), the standard form of parabolic cylinder equation, the limiting process $Pr \to \infty$ has been made. The solution to (36) and (37) can be obtained, in terms of η, as

$$\theta(\eta) = -Qe^{-2m\eta} + (1+Q)e^{-\frac{Pr\eta^2}{2}}\left[M\left(\frac{3-\alpha}{2},\frac{1}{2},\frac{Pr\eta^2}{2}\right) - \sqrt{2Pr}\frac{\Gamma\left(\frac{4-\alpha}{2}\right)}{\Gamma\left(\frac{3-\alpha}{2}\right)}\eta M\left(\frac{4-\alpha}{2},\frac{3}{2},\frac{Pr\eta^2}{2}\right)\right]. \quad (38)$$

The dimensionless surface temperature gradient is given by

$$\theta'(0) = 2mQ - (1+Q)\sqrt{2Pr}\frac{\Gamma\left(\frac{4-\alpha}{2}\right)}{\Gamma\left(\frac{3-\alpha}{2}\right)}. \quad (39)$$

Apparently, the combined parameter m has no effect on the heat transfer rate in the PST case for $Pr \to \infty$ when the Eckert number Ec vanishes or $Q = 0$.

(ii) The PHF case

For the limiting case $Pr \to \infty$, we have the solution in the PHF case as

$$g(\eta) = -Qe^{-2m\eta} + (1+2mQ)e^{-\frac{Pr\eta^2}{2}} \times \left[\sqrt{\frac{1}{2Pr}}\frac{\Gamma\left(\frac{3-\alpha}{2}\right)}{\Gamma\left(\frac{4-\alpha}{2}\right)}M\left(\frac{3-\alpha}{2},\frac{1}{2},\frac{Pr\eta^2}{2}\right) - \eta M\left(\frac{4-\alpha}{2},\frac{3}{2},\frac{Pr\eta^2}{2}\right)\right]. \quad (40)$$

Thus the dimensionless surface temperature can be obtained as

$$g(0) = -Q + (1+2mQ)\sqrt{\frac{1}{2Pr}}\frac{\Gamma\left(\frac{3-\alpha}{2}\right)}{\Gamma\left(\frac{4-\alpha}{2}\right)}. \quad (41)$$

Again, as in the PST case, the combined parameter m has no substantial effect on the surface temperature $g(0)$ in the PHF case when the Eckert number is zero or $Q = 0$ for $Pr \to \infty$.

4. Results and discussions

The flow and heat transfer in a laminar flow of an electrically conducting second grade fluid subject to a transverse uniform magnetic field over a stretching sheet with power-law surface temperature and power-law surface temperature gradient have been examined. The energy equation includes the viscous dissipation, work done due to deformation, internal heat generation or absorption and the Joule heating. The closed form solutions of the velocity components show that the combined parameter m depends on the viscoelastic parameter K of the second grade fluid as well as the magnetic parameter Mn. In Fig. 1(a) and (b), we plot the dimensionless velocity component $f'(\eta)$ as a function of η for several values of viscoelastic parameter K and magnetic parameter Mn, respectively. It can be observed that $f'(\eta)$ decreases with η for both K and Mn keeping constant. For a fixed position η, $f'(\eta)$ increases with K but decreases with Mn. Thus the viscoelasticity can increase the momentum boundary layer thickness while the presence of the magnetic field decreases it. It follows from (12) or Fig. 2, we know that the magnitude of dimensionless surface velocity gradient, $|f''(0)| = m$, increases with the magnetic parameter Mn but decreases with the viscoelastic parameter K. This implies that the role played by the viscoelasticity of the second grade fluid is to reduce the skin friction at the sheet, while the presence of magnetic field is to increase the power needed to stretch the sheet.

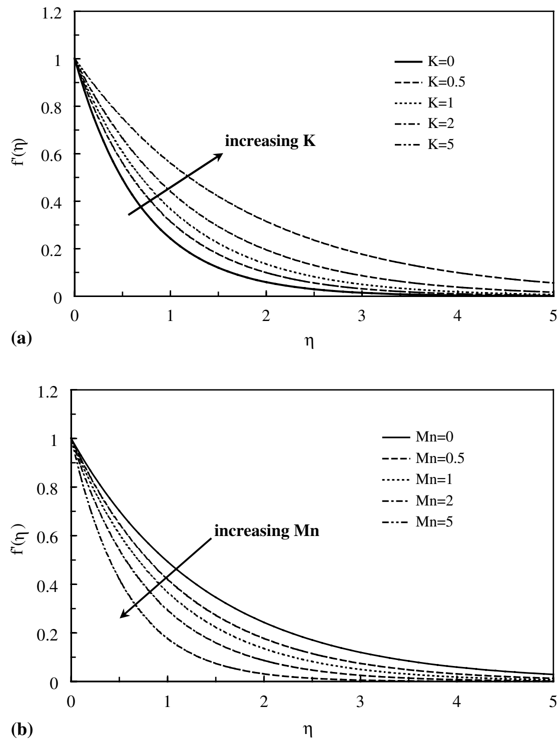


Fig. 1. (a) The velocity component $f'(\eta)$ for selected K and Mn = 1. (b) The velocity component $f'(\eta)$ for selected Mn and $K = 1$.

Our results of dimensionless surface temperature gradient in the PST case with selected parameters, listed in Table 1, are compared with those obtained by Vajravelu and Roper [9] using a numerical method to the same problem of a non-electrically conducting second grade fluid and the comparison is in a good agreement. Obviously, the effect of small Eckert number is to reduce

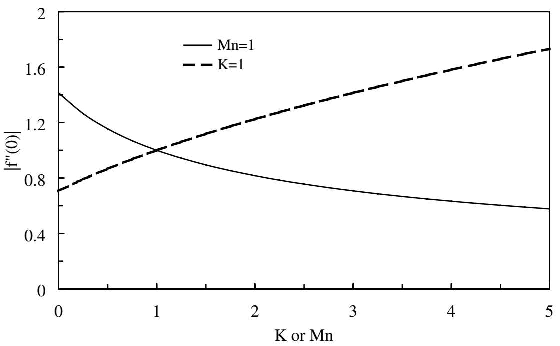


Fig. 2. The dimensionless wall shear stress $f''(0)$ vs. K or Mn.

the heat transfer rate (in absolute sense) from the wall to the fluid region in the PST case.

In the PST case we plot the dimensionless temperature profile $\theta(\eta)$, as shown in Fig. 3(a) and (b), for various values of K and Mn, respectively, while other parameters are fixed. For a given position η, $\theta(\eta)$ decreases as the viscoelastic parameter K increases, resulting in a decrease of the thermal boundary layer thickness. However, the decrease in thermal boundary layer thickness is not appreciable since the viscoelastic parameter is usually not large and it can be taken as a small perturbation parameter as done by Rajeswari and Rathna [19], Beard and Walters [20], etc. It can be observed in Fig. 3(b) that $\theta(\eta)$ increases with the magnetic parameter Mn at a given location η; therefore, the thermal boundary layer thickness will increase with Mn. The variations of dimensionless temperature distribution with K and Mn in the PHF case are similar to those in the PST case, as seen in Fig. 4(a) and (b). Since the magnitude of the increase of thermal boundary layer thickness due to the magnetic parameter Mn is more appreciable than that decreased due to the viscoelastic parameter K, we can expect that the thermal characteristics are more influenced by Mn than those by K in this problem.

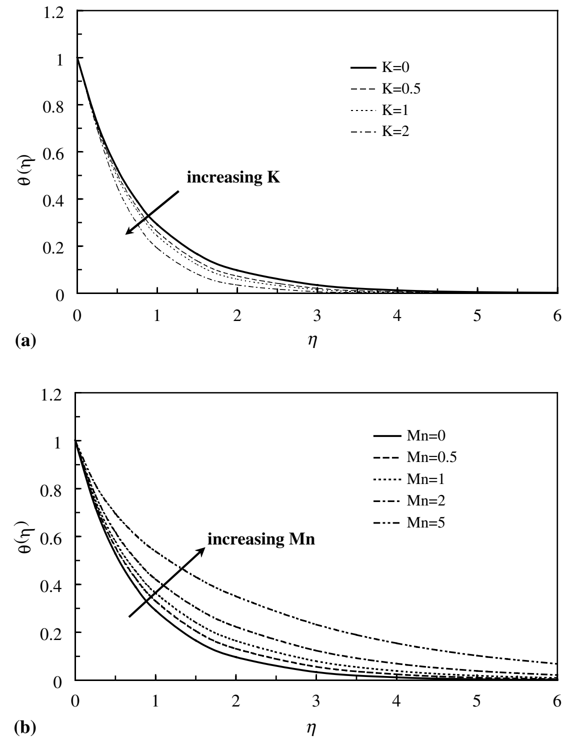
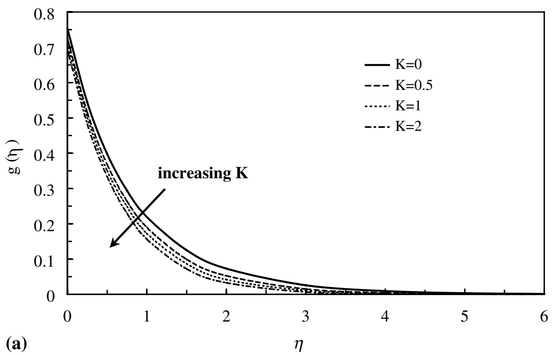


Fig. 3. (a) The dimensionless temperature profile (PST) $\theta(\eta)$ for various K and $\mathrm{Mn} = 0$, $\alpha = 0$ $Pr = 1$, $Ec = 0.01$. (b) The dimensionless temperature profile (PST) $\theta(\eta)$ for various Mn and $K = 0$, $\alpha = 0$ $Pr = 1$, $Ec = 0.01$.

The dimensionless surface temperature gradient $\theta'(0)$ in the PST case and dimensionless surface temperature $g(0)$ in the PHF case together with their asymptotic results are tabulated in Tables 2 and 3 for small $Ec = 0.2$ and a variety of parameters. Apparently, in the PST case, the larger K, the larger (in absolute sense) the magnitude of the surface temperature gradient when other parameters are fixed. This implies that the viscoelasticity of the second grade fluid will enhance the heat transfer rate

Table 1
Comparison of dimensionless temperature gradient $\theta'(0)$ in the PST case between numerical solution [9] and the present study for selected parameters

Parameters		$\theta'(0)$			
		Vajravelu and Roper [9]	Present study	Vajravelu and Roper [9]	Present study
$\mathrm{Mn} = 0$	Pr	$Ec = 0$		$Ec = 0.02$	
$K = 0$	1	−1.710937	−1.71094	−1.705156	−1.70516
$\alpha = -1$	5	−4.028535	−4.02854	−4.010094	−4.01010
$K = 1$	1	−1.757867	−1.75787	−1.750994	−1.75099
$\alpha = -1$	5	−4.079128	−4.07913	−4.088629	−4.05863
$K = 1$	1	−1.414214	−1.41421	−1.406186	−1.40619
$\alpha = -1$	5	−3.391900	−3.39190	−3.367179	−3.36718



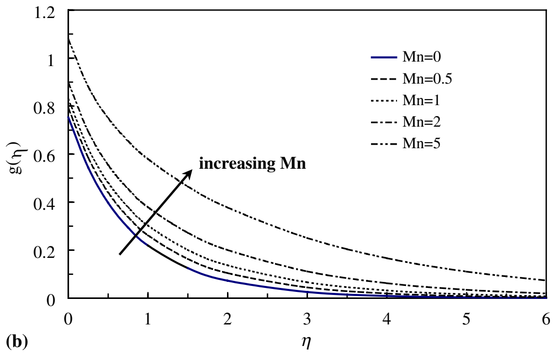


Fig. 4. (a) The dimensionless temperature profile (PHF) $g(\eta)$ for various K and $\mathrm{Mn}=0$, $\alpha=0$, $Pr=1$, $Ec=0.01$. (b) The dimensionless temperature profile (PHF) $g(\eta)$ for various Mn and $K=0$, $\alpha=0$, $Pr=1$, $Ec=0.01$.

from the wall to the fluid region. We have observed that the increase of the Prandtl number may also result in an increase of $|\theta'(0)|$ in the PST case. The magnetic parameter Mn and the internal heat parameter α may reduce the values of $|\theta'(0)|$ resulting in a decrease in the heat transfer rate. It should be noted that, as described in Subhas and Veena [15] and Vajravelu and Roper [9], the values of $\theta'(0)$ are all negative at least in the range of parameters they considered. Physically, this indicates that the heat flow is always transferred from the wall to the fluid. However, if the Eckert number is large enough, the heat transfer may reverse its direction, as shown in the last paragraph. From above paragraphs, we may conclude that the presence of magnetic field will inhibit the fluid motion in view of the decrease of the momentum boundary layer thickness and reduce the heat transfer from wall to the fluid. Conversely, the visco-elasticity of the second grade fluid may enhance both the fluid motion and the heat transfer rate at the wall.

As for the surface temperature in the PHF case, we observe that the increases in Mn and α will increase the wall temperature $g(0)$, whereas the increases in K and Pr will reduce it. The missing values in Tables 2 and 3 are due to the condition $4-2r+\alpha r\to 0$ is met. The asymptotic results in both the PST and PHF cases are in a good agreement when the Prandtl number is greater than 10. The maximum relative errors are no more than 5% at least for the parameters we considered in two cases. The larger the Prandtl number, the lesser the relative error.

Table 2
The comparison of exact solution (24) and asymptotic solution (39) of $\theta'(0)$, with selected parameters in the PST case

Parameters		$\theta'(0)$					
		$\alpha=-0.1$		$\alpha=0$		$\alpha=0.1$	
$Ec=0.2$	Pr	Eq. (24)	Eq. (39)	Eq. (24)	Eq. (39)	Eq. (24)	Eq. (39)
$\mathrm{Mn}=0$	1	−1.31371	−1.58698	−1.26667	−1.55535	−1.21562	−1.52320
$K=0$	10	−4.56184	−4.77311	−4.44726	−4.66548	−4.32933	−4.54484
	100	−14.6646	−14.8784	−14.3128	−14.5335	−13.9513	−14.1792
	500	−32.8665	−33.0801	−32.0805	−32.3007	−31.2725	−31.4999
$\mathrm{Mn}=0$	1	−1.37488	−1.20211	–	–	−1.29111	−1.86556
$K=1$	10	−4.59962	−4.75037	−4.48696	−4.64270	−4.37115	−4.53217
	100	−14.6843	−14.8354	−14.3328	−14.4887	−13.9715	−14.1325
	500	−32.8796	−33.0305	−32.0931	−32.2488	−31.2848	−31.4455
$\mathrm{Mn}=1$	1	−1.11691	−1.50407	−1.05451	−1.47250	−0.97415	−1.44036
$K=0$	10	−3.89105	−4.07466	−3.75932	−3.93735	−3.62247	−3.79182
	100	−12.0368	−12.2726	−11.6152	−11.8548	−11.1779	−11.4208
	500	−26.4720	−26.7099	−25.5050	−25.7468	−24.4999	−24.7543
$\mathrm{Mn}=1$	1	−1.18298	−1.50831	−1.13333	−1.47450	−1.07920	−1.44028
$K=1$	10	−3.87868	−4.03362	−3.74805	−3.90392	−3.61261	−3.76874
	100	−11.9411	−12.1088	−11.5144	−11.6849	−11.0715	−11.2445
	500	−26.3409	−26.5092	−25.3661	−25.5372	−24.3523	−24.5259

Table 3
The comparison of exact solution (32) and asymptotic solution (41) of $g(0)$, with selected parameters in the PHF case

Parameters		$g(0)$					
		$\alpha = -0.1$		$\alpha = 0$		$\alpha = 0.1$	
$Ec = 0.2$	Pr	Eq. (32)	Eq. (41)	Eq. (32)	Eq. (41)	Eq. (32)	Eq. (41)
Mn = 0	1	0.772523	0.639074	0.800000	0.651988	0.832052	0.665614
$K = 0$	10	0.273601	0.266340	0.281352	0.273625	0.289711	0.281545
	100	0.147368	0.146637	0.152698	0.151918	0.158526	0.157694
	500	0.117986	0.117843	0.122954	0.122799	0.128413	0.128248
Mn = 0	1	0.742096	0.875724	–	–	0.788006	0.446806
$K = 1$	10	0.276367	0.270762	0.284146	0.278139	0.292579	0.286127
	100	0.149833	0.149281	0.155319	0.154724	0.161312	0.160679
	500	0.119317	0.119208	0.124371	0.124254	0.129929	0.129804
Mn = 1	1	0.908300	0.690056	0.955167	0.703902	1.02267	0.718561
$K = 0$	10	0.397933	0.402150	0.411686	0.417917	0.426823	0.435756
	100	0.307067	0.306860	0.319939	0.319777	0.334090	0.333988
	500	0.293061	0.293014	0.306512	0.306473	0.321337	0.321307
Mn = 1	1	0.876313	0.687447	0.900000	0.702651	0.938308	0.718610
$K = 1$	10	0.412923	0.410131	0.427117	0.424542	0.442665	0.440421
	100	0.317311	0.316932	0.330801	0.330421	0.345635	0.345259
	500	0.298609	0.298532	0.312424	0.312347	0.327665	0.327597

Table 4
The results of $\theta'(0)$ and $g(0)$ with and without the inclusion of the Joule heating for various parameters with $K = 0$ and $\alpha = 0$

		$\theta'(0)$		$g(0)$	
		With Joule heating	Without Joule heating	With Joule heating	Without Joule heating
Ec = 0.01					
$Pr = 1$	Mn = 1	−1.20771	−1.21040	0.829154	0.826944
	Mn = 10	−0.744125	−0.758077	1.33083	1.31279
$Pr = 10$	Mn = 1	−4.64367	−4.65918	0.223134	0.219826
	Mn = 10	−3.95763	−4.06015	0.291231	0.266664
$Pr = 100$	Mn = 1	−15.4095	−15.4761	0.0768584	0.0725938
	Mn = 10	−13.9859	−14.5305	0.141692	0.105693
Ec = 0.1					
$Pr = 1$	Mn = 1	−1.13514	−1.16202	0.888844	0.866737
	Mn = 10	−0.480443	−0.619953	1.167177	1.49138
$Pr = 10$	Mn = 10	−4.22477	−4.37992	0.312448	0.279368
	Mn = 10	−2.02012	−3.04526	0.755538	0.509873
$Pr = 100$	Mn = 1	−13.6122	−14.2779	0.192002	0.149356
	Mn = 10	−3.69219	−9.1386	0.822059	0.462076

Now, we examine the effects of the inclusion of Joule heating in this boundary layer flow problem by comparing the corresponding values given in Table 4. When the Joule heating is taken into account, the values of $|\theta'(0)|$ in the PST case are reduced and the values of $g(0)$ in the PHF case are increased. The effects of Joule heating on the thermal characteristics at the wall are magnified when *Ec* becomes large in view of the relative differences between the corresponding values of $\theta'(0)$ and $g(0)$, respectively. Since the heat generated due to the Joule heating and the total heat, except for internal heat generation or absorption, produced in the fluid region are characterized by Mn and *Ec*, respectively. Thus the temperature in the fluid region can be raised to reduce the heat transfer rate from the wall to the fluid region in the PST case and to increase the surface temperature in the PHF case when *Ec* and/or Mn are large.

Since the previous authors [7,9,13,15] did not focus on the case when the amount of heat generated in the fluid region is large, i.e. the Eckert number is large, either they

Table 5
The thermal characteristics $\theta'(0)$ and $g(0)$ at the wall with Ec, Mn and Pr when $K = 1$, $\alpha = 0$

$\theta'(0)$		$Pr = 1$	$Pr = 10$	$Pr = 25$	$Pr = 100$	$Pr = 500$	$Pr = 1000$
Mn = 1	$Ec = 0.1$	−1.23333	−4.27246	−6.79699	−13.6132	−30.4019	−42.9708
	$Ec = 0.7$	−0.63333	−1.12598	−1.09012	−1.02057	−0.18712	0.512362
	$Ec = 0.8$	−0.53333	−0.60156	−0.25564	1.0782	4.84867	7.75956
	$Ec = 1.0$	−0.33333	0.44726	1.61331	5.27574	14.9203	22.254
	$Ec = 1.5$	0.16667	3.06933	6.2857	15.7696	40.0992	58.4899
Mn = 10	$Ec = 0.1$	−0.59708	−1.83961	−2.33051	−2.76938	−2.45123	−1.90274
	$Ec = 0.2$	−1.88413	0.763308	2.72643	9.83738	30.2036	46.0819
	$Ec = 0.3$	0.20226	3.36622	7.78337	22.4442	62.8584	94.0665
$g(0)$		$Pr = 1$	$Pr = 10$	$Pr = 25$	$Pr = 100$	$Pr = 500$	$Pr = 1000$
Mn = 1	$Ec = 0.1$	0.825	0.317799	0.250208	0.197224	0.170321	0.164228
	$Ec = 0.7$	1.275	0.973738	0.97541	0.998691	1.02294	1.03012
	$Ec = 0.8$	1.35	1.08306	1.09628	1.13227	1.16504	1.17443
	$Ec = 1.0$	1.5	1.30171	1.33801	1.39942	1. 44925	1.46306
	$Ec = 2.0$	2.25	2.39495	2.54668	2.7352	2.87028	2.90621
Mn = 10	$Ec = 0.1$	1.43405	0.811007	0.819896	0.884927	0.958662	0.981904
	$Ec = 0.2$	1.8369	1.39692	1.50443	1.70482	1.88884	1.94376
	$Ec = 0.3$	2.23976	1.98283	2.18896	2.52471	2.81901	2.90562

concluded that the heat flow always transfers from the wall to the fluid region or they implicitly showed this result in their figures or tables, or they did not consider the magnetic effects in this heat transfer problem. Some results regarding the effects of Ec, Mn and Pr on the thermal characteristics $\theta'(0)$ and $g(0)$ are listed in Table 5. We observed that when Ec is small, the value of $|\theta'(0)|$ is negative and decreases more negatively with Pr, implying that the heat transfer rate from wall to the fluid increases with Pr since the thermal boundary layer thickness decreases with Pr. When Ec is large enough, for example $Ec = 1.0$ and $\mathrm{Mn} = 1$, the value of $\theta'(0)$ is -0.333333 at $Pr = 1$ and increases to 22.254 at $Pr = 1000$. However, $\theta'(0)$ becomes positive at $Pr = 1$ and increases more positively for $Ec = 1.5$. This phenomenon is mainly due to the fact that the amount of heat generated in the fluid region is large enough so that the fluid temperature near the wall is higher than that at the wall, resulting in a heat flow from the fluid region to the wall ($\theta'(0) > 0$). This situation is more magnified when the Prandtl number is large because the thermal boundary layer thickness will be compressed ($\delta_T \sim 1/\sqrt{Pr}$). Thus, it can be expected that when Ec is intermediate, for example $Ec = 0.7$, the value of $\theta'(0)$ is negative first and then decreases to a minimum and finally increases to a positive value with Pr, as shown in Table 5. This behavior can also be observed for $\mathrm{Mn} = 10$, which corresponds to more heat generated in the fluid region, therefore, $\theta'(0)$ becomes a positive value at about $Ec = 0.3$ rather than at about $Ec = 1.5$ for $\mathrm{Mn} = 1$. Apparently the surface temperature $g(0)$ in the PHF case will increase with Ec irrespective of Pr according to our physical argument. It is, therefore, straightforward that both the values of $\theta'(0)$ and $g(0)$ will increase with the internal heat parameter α and magnetic parameter Mn. The above results are similar to those of Liu [21] for the same problem of a conducting viscoelastic fluid.

5. Conclusions

The velocity profile is strongly dependent on the viscoelastic parameter K as well as the magnetic parameter Mn. In the PST case, the increases of K and Pr will result in an increase in $|(\theta'(0)|$ when Ec is small, however, the increases of Mn and α will decrease it. When Ec is not small, the heat may flow from the fluid region to the wall in contrast to the case when Ec is small. A physical argument is proposed for this behavior. The values of $g(0)$ in the PHF case will decrease as the parameters K and Pr increase, and will increase as Mn and α increase. The inclusion of the Joule heating could significantly influence the thermal characteristics at the wall than that without it. Beyond the Prandtl number 10, the asymptotic solutions give good approximations of the thermal characteristics at the wall for $Ec = 0.2$ within the relative error 5%. The larger the Prandtl number, the lesser the relative error.

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